Logic and Probability

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As is clear from the other articles in this volume, logic has applications in a broad range of areas of philosophy. If logic is taken to include the mathematical disciplines of set theory, model theory, proof theory, and recursion theory (as well as first-order logic, second-order logic, and modal logic), then the only other area of mathematics with such wide-ranging applications in philosophy is probability theory.

Perhaps most centrally, probability is important in epistemology and the philosophy of mind as a way to represent "degrees of belief" in between (subjective) certainty of truth and (subjective) certainty of falsehood. As a result, it has great importance for the philosophy of science, in understanding how scientists respond to evidence for or against their theories. This has led to the tradition known as "Bayesianism" in epistemology, the philosophy of science, statistics, and related fields. These applications will be discussed throughout this article.

The notion of probability as a degree of belief has also had major impact in fields related to ethics and action. One traditional view of motivation, as discussed by David Hume, holds that our choices are shaped by our beliefs and desires, working together. One modern formulation of this view is given by the notion of "expected value" in decision theory (as discussed in section 2.4). This has shaped the theory of action and philosophical discussion of free will, and has also affected ethical theories like utilitarianism. Where Jeremy Bentham's original formulation of utilitarianism says that one ought to do what results in the greatest good for the greatest number of people, contemporary discussions generally accept that in cases where one is uncertain how various actions will affect people, one ought to do what results in the greatest *expected* good, in the technical sense of "expectation" from decision theory.

But there are also important applications of probability in other areas of philosophy. The notion of a "chance", understood as something like an objective tendency of a certain type of non-deterministic situation to produce a relevant outcome, is very important in metaphysics and the philosophy of science. It has contributed to the debate in metaphysics about the notion of causation ((Spirtes et al. 2000), (Pearl 2000), (Woodward 2003), (Hitchcock 2010)) and also to discussions in philosophy of language about counterfactual conditionals (Gundersen 2004). Since this interpretation of probability has not traditionally had as much connection to logic as others, I will have relatively little to say about it.

Finally, as I will discuss in sections 3 and 4, some philosophers have also used various notions of probability to help clarify our understanding of logic itself.

1 Similarities between probability and logic

One feature shared by logic and probability that is probably central to why they are the two fields of mathematics with extensive philosophical application, is their discussion of sentences. While all areas of mathematics *use* sentences to discuss their subject matters, only in logic and probability are the sentences themselves the *object* of study. Logic is concerned primarily with an entailment relation that holds between sentences or sets of sentences. Probability is concerned primarily with a function that assigns real numbers to individual sentences.

Both subjects emerged as part of the study of reasoning. Logic is often described as the science of correct reasoning. This can't be quite right, for at least two reasons. First, as Gilbert Harman pointed out (Harman 1986, Ch. 1), if one believes certain premises, and learns of a valid argument from these premises to some conclusion, one doesn't therefore have to believe the conclusion. Depending on the conclusion, one might instead choose to reject one of the premises. The relation of deductive logical entailment certainly puts some constraints on our beliefs, but it is not itself the relation of correct inference.

The other reason why logic is not the science of correct reasoning is because it leaves out many perfectly good inferences. When one sees fresh tracks in the mud, it is perfectly reasonable to conclude that someone must have walked by recently, even though there are clearly many other logically consistent ways for the tracks to have ended up there. Probability theory is often taken to be a way to help spell out these inferences that are not deductively valid, but nevertheless seem good. (There are of course other alternatives that have been proposed, such as default reasoning (Reiter 1980), belief revision (Alchourròn et al. 1985), (Darwiche & Pearl 1997), formal learning theory (Kelly 1996), statistical learning theory (Vapnik 1989), and various systems called "imprecise probability theory" (Walley 1991). These are beyond the scope of this article, but several of them do have interesting connections to probability.) But before I go into the details of how probability is supposed to help fill this gap left by deductive logic, it will be helpful to explain the mathematical foundations of probability.

2 Foundations of probability

2.1 The axioms

Mathematical discussions of probability theory traditionally begin with the formal system laid down in (Kolmogorov 1950). First we begin with a set Ω and a function P that assigns a real number to each subset of Ω.[1]

1. $P(\Omega) = 1$
2. If $A \subseteq B$ then $P(A) \leq P(B)$
3. If A and B are disjoint (they have no elements in common), then $P(A \cup B) = P(A) + P(B)$

If Ω is finite, then these axioms give a way to characterize probability functions. If P satisfies the axioms, then we only need to know its values on the singletons to characterize its value on every set—$P(A)$ will equal the sum of $P(\{a\})$ for all $a \in A$. And in fact, we don't need to specify the value on every singleton—since the values must add up to 1, we can specify the values on all but one of them.

Of course, as I said earlier, the objects of probability are standardly taken not to be sets, but rather sentences, like the objects of logic. So these axioms need some interpretation in order to be useful. It is standard to take Ω to be something like the set of all "possibilities" (this will be clarified further in section 2.2), and then to say that the probability of any sentence is $P(A)$, where A is the set of possibilities on which that sentence is true.

But other formulations of probability theory are available that make P directly a function of sentences, rather than sets. For instance, consider the following axioms:

1. $P(A) = 1$ if A is a tautology
2. If $A \vdash B$ then $P(A) \leq P(B)$

[1] If Ω is infinite, then it is standard not to assume that P assigns probabilities to *every* subset, but just to a particular collection $\mathcal{F}$ of them. Kolmogorov assumes that $\Omega \in \mathcal{F}$, that the complement of any set in $\mathcal{F}$ is in $\mathcal{F}$, and that the union of any countable collection of elements of $\mathcal{F}$ is in $\mathcal{F}$. Axiom 3 is also extended to apply to infinite collections of pairwise disjoint sets, as well as pairs of disjoint sets.

3. If A and B are logically incompatible, then $P(A \vee B) = P(A) + P(B)$

As you can see, these axioms are very similar to the axioms endorsed by Kolmogorov. If we assume that the set Ω is some finite set of models in some first-order language, and associate sentences with the set of models in which they are true, then the two sets of axioms are equivalent.

However, the two different formulations are useful because of the different modifications and interpretations they permit. For instance, the set-based formulation lends itself better to dealing with the infinite. As mentioned in footnote 1, mathematicians traditionally extend Kolmogorov's third axiom to apply to the union of countably many pairwise disjoint sets as well as to finite unions. If we are working in a formal language that doesn't include a device of infinitary disjunction, then it is difficult to add a corresponding axiom to the sentence-based formulation. But as it turns out, this axiom of countable additivity is needed to prove several important mathematical theorems about probability theory, like the Central Limit Theorem and the Laws of Large Numbers.[2] Thus, mathematicians generally prefer the set-based formulation of the axioms.

However, for many philosophical applications, the sentence-based formulation is more useful. This will become especially clear in sections 4 and 5, where logic is developed from probability theory, or non-classical logics replace the classical notions of "tautology", "$\vdash$" and "logically incompatible" used in the three axioms. But other advantages of this formulation will show up throughout the article, so from here on, I will assume that the objects of the probability function are sentences, except where otherwise mentioned.

2.2 Interpretations of probability

At this point, I have given the formal mathematical characterization of probability, but this says very little about what the function P is supposed to represent. The project of giving the meaning of this function is traditionally called the project of "interpreting probability". I will just give a

[2]These theorems all concern the behavior of sums of many independent random variables. They show that even though the individual variables may have very strange and complicated probabilities of taking on various values, the behavior of the sum of a large number of them can be known with high accuracy using just two parameters derived from the distributions of the individual variables. These theorems are of great importance in mathematical probability theory and statistical inference.

brief outline of a few of the most important interpretations here; readers interested in more details should consult (Hájek 2009).

Perhaps the most intuitive notion of probability is the notion of chance, which has already been mentioned in the introduction. We want to say that the chance of a normal coin coming up heads on its next flip is $1/2$, but that a biased coin may have a higher or lower chance of coming up heads on any given flip. Additionally, some of our scientific theories (especially quantum mechanics, but perhaps also statistical mechanics) seem to tell us that the world behaves in fundamentally non-deterministic ways that are nevertheless governed by chances.

These chances are supposed to be objective aspects of the physical world, independent of what anyone knows or believes. But many philosophers, scientists, and statisticians have been suspicious of this notion of chance, because it seems very hard to say exactly what it means. In particular cases, all we observe is that an event did or didn't happen; we don't observe how likely it is that it *would* have happened. And of course, this counterfactual aspect to chance is part of what makes it so important in understanding the notions of risk, causation, and many other things.

One way to make this notion of chance clearer is to relate it to frequencies. When we say that a particular coin is biased so that it has a $2/3$ chance of landing heads, perhaps this means that out of all the times that the coin is flipped, it comes up heads $2/3$ of them. When we say that an electron with measured spin up in one direction has a $1/3$ chance of being observed with spin up when measured in a direction at angle 60 degrees from the first direction, perhaps this means that if we perform this experiment a large number of times, then $1/3$ of the electrons will be observed to agree with this.

This interpretation of chances as frequencies has been quite influential in the discipline of statistics. It is straightforwardly well-defined in most cases of interest, and only refers to notions that are scientifically respectable. Additionally, it is clear that frequencies satisfy the Kolmogorov axioms of probability—just let Ω be the set of all cases of the type we are interested in, and then say that every individual case has probability $1/||\Omega||$, where "$||\Omega||$" refers to the number of elements of Ω.

However, there are some clear problems that arise in using frequencies to analyze the notion of chance. For instance, a coin that is flipped exactly 101 times in its existence cannot possibly have exactly half of its flips come up heads and exactly half tails. Thus, saying that chance *is* frequency means that a coin that is flipped exactly 101 times in its existence cannot possibly be fair, which seems strange. Even worse, certain types of events that we intuitively want to ascribe chances to are essentially unrepeatable. For instance, we might be interested in the chance that a large earthquake will strike Gujarat in the next 30 years. "The next 30 years" refers to a specific period in time—at best, the frequency theory can tell us to ask how many distinct 30 year periods are such that Gujarat experiences a large earthquake. But intuitively, it seems that the tectonic stresses might change over time, so that periods in the distant future or past would be irrelevant to the chance over the next 30 years. Similarly, some scientific theory might tell us something about the chance of the universe coalescing into a single large black hole during the first few seconds of its existence. Since there is no way to repeat the first few seconds of existence of the universe, it seems that the frequency must be 0 (after all, the universe did not actually coalesce into a single black hole early on), so that if the chance was non-zero, then frequency and chance must be distinct things.

Nevertheless, frequencies can be quite useful in understanding chance. For many types of processes, such as drawing lottery tickets out of a hat, it is reasonable to model each ticket as having an equal chance of being drawn, so that the chance of a particular person winning just is the fraction of tickets she owns, as suggested by the frequency interpretation. And frequencies are quite useful for applications of probability theory in other areas of mathematics as well.

In stark contrast to the chance and frequency interpretations, there is the Bayesian interpretation of probability as something like "rational degree of belief". While there may be strange metaphysical commitments involved in discussing a notion of chance, it seems very plausible that an agent might believe something to a greater or lesser degree. And of course, these notions can definitely coexist. Even if a coin is objectively biased towards heads, I might be uncertain of how it is biased, and therefore I might rationally have no stronger belief that it will come up heads on the next flip than that it will come up tails.

Of course, just given the fact that an agent believes one thing to a greater degree than another, it is not immediately obvious how to associate these degrees of belief with precise numerical values, as needed to provide an interpretation of probability. However, defenders of this interpretation have a clever argument. (Ramsey 1926) (de Finetti 1937)

They begin with the assumption that we can measure someone's degree of belief in a proposition by seeing how much that person considers to be a fair price for a bet that returns Rs 100 if the proposition is true. Someone who considers Rs 80 fair is said to have degree of belief 0.8, while someone who considers Rs 20 fair is said to have degree of belief 0.2. There are then two further assumptions. First, the person is assumed to be willing to buy a bet for any price less than the price she considers fair and to sell it for any price greater than the price she considers fair. Thus, there is no spread between her prices on these bets, unlike the prices listed in real casinos. Second, if a person is willing to accept certain bets independently, then she is assumed to be willing to accept them in combination as well.

Together, these assumptions show that if an agent violates the probability axioms, then there is a combination of bets she is willing to accept that guarantee she will lose money. For instance, if someone's degree of belief in a tautology is greater than 1, then she is willing to pay more than 100 Rs for a bet that can only yield Rs 100, while if her degree of belief in a tautology is less than 1, then she is willing to sell a bet on it for less than Rs 100—since the proposition is a tautology, she is guaranteed to have to pay out Rs 100 and thus lose money. Similarly, if $A \vdash B$ and someone has a higher degree of belief in A than in B, then she is willing to sell a bet on B for a certain amount and buy a bet on A for a greater amount. Thus, she loses money in the buying and selling of bets, and can only break even or come out ahead if she wins both bets. But since A entails B, it is impossible for her to win both, so she is guaranteed to lose money. Finally, if A and B are logically incompatible, but her degree of belief in $A \vee B$ is greater than the sum of her degrees of belief in A and B, then she is willing to buy a bet on $A \vee B$ and sell bets on A and B individually for a loss. If she wins the bet on $A \vee B$, then she must lose one of the other two bets, and the payoffs cancel out, and otherwise there are no payouts of any of the bets, so she loses money overall in any case.

Thus, if we assume that a rational agent will never be willing to enter into a collection of bets where she is guaranteed to lose, then we see that a rational agent whose betting behavior satisfies the assumptions of this argument must have degrees of belief that satisfy the probability axioms. Additionally, one can show that any agent whose degrees of belief satisfy these axioms is safe from this situation of a guaranteed loss. (For obscure historical reasons, this collection of bets with a guaranteed loss is known as a "Dutch book", and the two results mentioned here are known as the "Dutch book theorem".)

The assumptions of this argument are rather strong, and have struck some authors as implausible. But there are several other arguments with the same conclusion—a rational agent must have degrees of belief that obey the probability axioms. These include the argument from representation theorems (Savage 1954) (Jeffrey 1965), the argument from Cox's theorem (Jaynes 2003, Chs. 1-2), and the argument from inaccuracy domination

(Joyce 1998), among others.

However, there are also some worries about the strength of the conclusion. In particular, the logical aspect of each of these axioms has worried many authors. Note that these axioms seem to require rational agents to be "logically omniscient". They seem to require that a rational agent recognize all tautologies, recognize whenever one sentence is a logical consequence of another, and recognize whenever two sentences are incompatible. This has struck some as an unreasonably demanding notion of rationality.

Thus, Ian Hacking, in (Hacking 1967), argues that these axioms should be slightly relaxed. In particular, he suggests that instead of requiring that rational agents have degree of belief 1 in any tautology, he requires only that they have degree of belief 1 in any proposition they *know* to be a tautology. Similarly, he replaces logical consequence and incompatibility with *known* logical consequence and incompatibility. (This is one of the ways that the sentence formulation is more easily modified than the set-theoretic formulation.) He thus also modifies the Dutch book argument slightly—instead of assuming that a rational agent won't accept bets that collectively guarantee that she will lose money, he only assumes that a rational agent won't accept bets that she *knows* will collectively guarantee that she will lose money.

This modification to the theory is relatively easy to make, but unfortunately it makes the theory much more difficult to work with mathematically. Therefore, other attempts to modify the requirement of logical omniscience have been proposed (Garber 1983), (Gaifman 2004), but there is still much controversy in this area.

2.3 Conditional probability

In addition to this notion of probability, there is also a notion of *conditional* probability. On the purely mathematical formulations of probability, conditional probability is sometimes taken to be a defined notion, with $P(A|B) = P(A \wedge B)/P(B)$, when $P(B) \neq 0$. However, given the various interpretations of probability, there are also corresponding interpretations of conditional probability. On the degree of belief interpretation, $P(A|B)$ is often taken to be something like the degree of belief that one would have in A if one were to learn that B is true. When thought of this way, the previously mentioned ratio formula is perhaps better replaced by a fourth axiom for probability:

4. $P(A|B)P(B) = P(A \wedge B)$

As pointed out in (Hájek 2003), this allows for conditional probabilities to be defined even in cases where $P(B) = 0$ (in which case the axiom provides no constraints beyond the requirement that $P(A \wedge B) = 0$, which was already required by axiom 2 from above), and also allows this axiom to be considered as a requirement on degrees of belief that someone could meet or fail to meet.

Given the role of conditional probability in representing a rational agent's beliefs on learning something new, it is natural to use conditional probability to measure the effect of various pieces of evidence. This gives probability a role in understanding reasoning as a diachronic process, taking place across time, which is a role that has sometimes been suggested for logic.

This project is known as "Bayesianism", and has been highly influential in the philosophy of science (Howson & Urbach 1989), as well as in epistemology (Bovens & Hartmann 2003) and statistics (Bernardo & Smith 2000). The idea is that an agent (for instance, a scientist) has a probability function recording her degrees of belief in every proposition (including competing scientific theories). As she learns more evidence, she updates her beliefs by replacing them with the conditional probabilities on the evidence she has learned. If the evidence E logically entails some hypothesis H, then the axioms of probability theory guarantee that $P(H|E) = 1$, and if the evidence contradicts the hypothesis, then $P(H|E) = 0$. However, the probabilities also give us a way to measure the force of the evidence in cases beyond logical entailment.

The basic idea of confirmation theory is that evidence E confirms a hypothesis H iff $P(H|E) > P(H)$, so that learning the evidence would make the agent more confident of the hypothesis. From the axioms of probability theory, we can prove (assuming that nothing relevant has probability 0 or 1) that this happens iff $P(E|H) > P(E)$.

Some authors have further suggested that we should be able to *measure* the amount of confirmation by using the probabilities. For instance, one might measure the degree of confirmation by the difference $P(H|E) - P(H)$, the amount that the evidence increases one's confidence in the hypothesis. This has the problem that a hypothesis that is already strongly supported can't receive very much more confirmation from new evidence. Other proposed measures have included the ratio $P(H|E)/P(H)$ (which is equal to $P(E|H)/P(E)$), and the "likelihood ratio", $P(E|H)/P(E|\neg H)$. And some authors have suggested that this plurality of measures gives us reason to believe that there is a plurality of notions of confirmation. (Fitelson 1999), (Christensen 1999), (Eells & Fitelson 2000), (Joyce 2005)

At any rate, with this probabilistic notion of confirmation, philosophers of science have set out to explain many historical patterns in scientific reasoning. For instance, one can show that if H entails E, then $P(E|H) = 1 \geq P(E)$. Since E confirms H iff $P(E|H) > P(E)$, we see that consequences of a hypothesis always confirm it unless they were already very strongly believed. Additionally, some philosophers have shown that surprising evidence is better than evidence that was already expected, and that having a variety of sources of evidence is more useful than many pieces of evidence from a single source. (Howson & Urbach 1989, Ch. 4)

Of course there are some problems with the Bayesian framework. Several of these problems are discussed in (Glymour 1981). The biggest one is probably the one called "the problem of the priors". This problem points out that in cases where there is no logical entailment, there will always be many different probability functions satisfying the axioms, and in particular, some of these probability functions will be such that E confirms H, while on others, E will actually *dis*confirm H. If the probability functions just represent the prior degrees of belief of various scientists, then this means that there will be no consensus about whether given evidence supports a given theory unless there is some degree of consensus about the priors.

One response, from the group known as "objective Bayesians", is to say that the probability axioms are not the only constraints on rational belief. Instead, there is a unique probability function that represents the rational beliefs of any agent with a given body of evidence. Then, the only disagreements about confirmation will be due to people having different evidence, which is no surprise. (Most Bayesian statisticians adopt something like this view.) (Jaynes 2003), (Wheeler & Williamson 2010)

Some subjective Bayesians choose to "bite the bullet" and accept that confirmation depends on subjective facts about the prior degrees of belief that different scientists have. But they point to results showing that given *enough* shared observations about a shared world, people with different prior probabilities will eventually converge to very similar probability distributions, which explains why in practice scientists generally agree about whether given pieces of evidence confirm particular theories. Of course, there is widespread disagreement about whether the amount of evidence needed for these results is actually available in practice. But they may also follow (Howson & Urbach 1989, p. 265) and just state that there is no problem with this subjectivity. "The situation is wholly analogous to deductive logic, where the logic is the inference engine: you choose the premises, and the engine generates the valid conclusions from them."

2.4 Decision theory

Decision theory is a generalization of the notion of betting mentioned above, which makes use of the notion of probability as degree of belief to update the traditional view of decisions as based on desires and beliefs. The fundamental concept of decision theory is the notion of "expected value". If the agent is uncertain about what outcomes will result from her action, then she may not directly know how good or bad this action is. But if she has numerical values for all of the possible outcomes, then she can take the average value of these outcomes, weighted by their probabilities (her degree of belief that they will happen if she performs the action) to give a value for the action. More formally, if an action has finitely many possible outcomes, then the expected value of the action is given by the sum over all possible outcomes O, of $P(O)V(O)$, where $P(O)$ is the degree of belief the agent has that this outcome will happen, and $V(O)$ is the value of that outcome. Decision theory then says that an agent ought to prefer the action with greatest expected value when making a decision. For more see (Savage 1954), (Jeffrey 1965), and (Joyce 1999).

3 Deriving probability from logic

Now that I have discussed the basic terms of probability, and given the basic interpretations, I will mention more how it relates to logic. As mentioned above, on most standard interpretations, the axioms of probability theory require a certain correspondence with deductive logic. But since probability plays a role in confirmation that generalizes the role of logical inference, some philosophers have thought that probability might be able to generalize deductive logic more explicitly.

Probably the most important figure in this discussion is Rudolf Carnap, whose *Logical Foundations of Probability* tried to develop this notion fully. (Carnap 1950) The basic idea is to start with the observation that if A entails B, then $P(B|A) = 1$, and if A contradicts B, then $P(B|A) = 0$. However, if we can make sense of a notion of *inductive* logic, then we may hope for some notion that tells us about a sort of *partial* entailment of one proposition by another. Carnap argued that a special probability function might be able to play this role. Unlike the Bayesian notion of probability, this probability function is unique, and has nothing to do with the actual degrees of belief of any agent. Unlike the frequency and chance interpretations, this function takes its values necessarily, and has nothing to do with the actual facts in the physical world.

Carnap's thought was as follows. Assume that the language we are using has predicates $A_1, A_2, \ldots, A_n$, and constants $c_1, c_2, \ldots, c_m$, and no quantifiers, and no other non-logical vocabulary. Then there are $m \cdot n$ atomic sentences that can be written in this language, and stating the truth-values of these atomic sentences will suffice to determine the truth-values of all sentences in the language. Thus, there are $2^{m \cdot n}$ ways that the world might be correctly described by this language. We call these conjunctions of atomic sentences and their negations, "state descriptions".

At first, Carnap suggested that a fully logical probability theory would assign each state description an equal probability. Then $P(A|B)$ would indicate the proportion of state-descriptions on which B was true, where A was also true. This seems to capture an intuitive notion of "partial entailment" that we might have.

However, this first function makes the notion of logical probability too weak for confirmation theory. We can calculate that no matter what evidence E might say about constants $c_1, \ldots, c_{m-1}$, it must be the case that $P(A_1 c_m|E) = 1/2$. If we imagine that $E = A_1 c_1 \wedge A_1 c_2 \wedge \cdots \wedge A_1 c_{m-1}$, then this means that the notion of probability seems not to respect induction. No matter how many objects we observe with property A_1, we can't conclude anything about whether some new object will have this property.

Thus, Carnap modified this function slightly. He said that two state-descriptions have the same "structure" if there is some way to permute the individual constants that will turn one state description into the other. If the language only has one predicate, this is equivalent to saying that two state-descriptions have the same structure iff they both attribute this predicate to the same number of objects. When there are multiple predicates, the requirement is that for each combination of predicates, the same number of objects are said to have this combination by each of the state-descriptions. Then, he defined a "structure-description" to be the disjunction of all state-descriptions that have the same structure.

The new probability function assigns equal probability to each structure-description, and then assigns each state-description with the same structure an equal probability. Thus, the state-descriptions themselves don't get equal probability. To take a very simple example, imagine a language with one predicate and three objects. There are four structure-descriptions (saying that no objects have the property, that one object does, that two objects do, and that all three objects do). Two of these structure-descriptions consist of only a single state-description, while the other two consist of three. Thus, the state-descriptions "$Ac_1 \wedge Ac_2 \wedge Ac_3$" and "$\neg Ac_1 \wedge \neg Ac_2 \wedge \neg Ac_3$" both get probability $1/4$, while all other state-descriptions get probability $1/12$.

We can thus calculate that $P(Ac_1) = 0 + 1/12 + 2/12 + 1/4 = 1/2$ (because it consists of no state-descriptions with the first structure, one with the second, two with the third, and one with the fourth), while $P(Ac_1 \wedge Ac_2) = 0 + 0 + 1/12 + 1/4 = 1/3$ (since it consists of only one state-description of the third structure, and the one with the fourth). Thus, $P(Ac_2|Ac_1) = (1/3)/(1/2) = 2/3$, and thus we see that one object having a property confirms the claim that another object does. As it turns out, this is the value of $P(Ac_2|Ac_1)$ no matter how many predicates and constants there are in the language. It may seem somewhat surprising that an individual observation of one object having a particular property can be so relevant to the same property of another object, but this is what Carnap's notion of logical probability predicts.

Carnap worried that while the first function doesn't give any confirmation by induction, this second function seems to give too much. So he introduced a parameter λ into the definition of the logical probability function that calibrated how much confirmation there would be from induction. He also tried to introduce modifications of the function to account for properties that come in families (where it is a logical requirement that every object satisfy exactly one property from each family). As an example of a family, we might consider colors, or any property that corresponds to the value of some parameter. These families were to deal with the worry that his function seems to say that the probability of "x is white" is $1/2$—but so is the probability of "x is yellow" and "x is maroon".

However, with these modifications, he lost some of the seemingly logical appeal of the probability function. The parameter λ allowed too much free choice, and there didn't seem to be a way to settle on one value for this parameter as the uniquely correct one for logic. And once there were families of properties, his probability function turned out to be language-sensitive. That is, there are two languages that can be taken to be expressively equivalent (in that every sentence of one language can be translated to an equivalent sentence of the other) and yet Carnap's procedure gives different probabilities to a sentence and its translation, because the properties are grouped into families in different ways.

Most philosophers have since abandoned the project of giving a purely logical probability function, and embraced something like the objective Bayesian program mentioned above if they think there should be some objectively correct answer to the amount that some evidence confirms a hypothesis. However, there are a few contemporary philosophers that defend something like Carnap's logical notion of probability.

In particular, Patrick Maher has been defending a view somewhat like Carnap's. Instead of deriving the values of the probability function from the syntactic form of the sentences however, Maher gives a slightly different characterization of what it means for the probability function to have its values as a matter of logic, and defends it against the criticisms of language-sensitivity. (Maher 2011) And Branden Fitelson has argued that we can still think of the notion of probabilistic confirmation as a logical notion, even if we use a probability function that is not itself logical in nature. (Fitelson 2006)

4 Deriving logic from probability

In addition to the program of finding a probability function that somehow generalizes the notions of logic, there have been several other programs aimed at deriving the notions of logic from probability theory.

4.1 Popper

Karl Popper was no fan of the idea of using subjective probability to measure the degree of confirmation of a theory by evidence. (In particular, he thought that evidence *never* confirms a theory. (Popper & Miller 1983)) However, he did think there were some important roles for probability to play in science, both in understanding chances and in some sense more akin to either the logical or degree of belief notions. But because he thought there was an important role for events of probability 0, he was unhappy with the standard formulation of probability that defines conditional probability in terms of unconditional probability. Thus, he gave a new axiomatization on which conditional probability is the basic notion. (Popper 1959), (Popper 1955)

In modern notation, these axioms are roughly as follows:

1. If $P(A|C) = P(B|C)$ for all C, then $P(D|A) = P(D|B)$ for all D.
2. $P(A|A) = P(B|B)$.
3. $P(A \wedge B|C) \leq P(A|C)$.

4. $P(A \wedge B|C) = P(A|B \wedge C)P(B|C)$.

5. $P(A|B) + P(\neg A|B) = P(B|B)$ unless $P(C|B) = P(B|B)$ for all C.

He then shows that if we let T be any tautology, and define $P(A) = P(A|T)$, then we get all the standard probability axioms as consequences of these, but that this system provides more constraints on conditional probabilities, in cases where the unconditional probability is 0.

However, before doing this, Popper notes a very interesting fact—at no point in this axiomatization did he make use of any of the logical properties of the symbols "$\wedge$" and "$\neg$". If we treat them as purely undefined symbols, then we can actually prove that they obey the standard rules of logic. That is, if ϕ and ψ are two strings built up from atomic letters using "$\wedge$" and "$\neg$" in such a way that they would be tautologically equivalent using the standard meanings of those symbols, then we can prove that $P(\phi|C) = P(\psi|C)$ for all C (and then the first axiom guarantees that $P(C|\phi) = P(C|\psi)$ for all C as well). Additionally, we can prove their intersubstitutability in complex expressions.

This is what Popper meant in the later paper when he described this (and the slightly different axiom system defined there) as "autonomous axiom systems"—these are axiomatizations of probability that don't depend on a prior understanding of logic for the object language. While the standard probability axioms require that we have a prior notion of logical consequence to use, these axioms define the notion of logical consequence at the same time as they define the notion of probability. (Of course, to use any axiom system, we need a notion of logical consequence for the metalanguage in which the system is phrased. But Popper shows that we don't first need to set up a notion of consequence for the object language as well.)

Popper treats this autonomy of the axiom system for probability as an interesting feature, but not an especially important one philosophically. However, Hartry Field, in (Field 1977), thinks that it suggests a more important idea. He considers some worries that Michael Dummett has about classical logic. (These worries are much deeper than I can do justice to in this brief summary, but they are discussed in just about all of Dummett's work about logic, semantics, or truth. In particular, see (Dummett 1973), (Dummett 1975), and (Dummett 1977).) In particular, Dummett argues that the notion of truth (in the standard sense in which we can say that every statement is either true or false) is too powerful for us to be able to grasp with our limited epistemic means. Thus, we shouldn't think of logical consequence as being a notion defined in terms of necessary truth-preservation. Instead, we should think of logical consequence as a relation that holds between sentences when we have a way of verifying one given a verification of the other. The meanings of statements come down to the roles they play in our inferential practices, rather than coming down to their truth-conditions as standardly thought.

One consequence of Dummett's claims is that classical logic should not be thought of as valid. Instead, we should only accept the rules of intuitionist logic. This consequence has been discussed by many philosophers of logic responding to Dummett. But Field makes use of Popper's autonomous formulation of the axioms to give an interesting alternative response.

Field's point is that probability can be used to understand degree of belief and confirmation, and thus in some way characterizes the inferential role of a proposition. (Note that this is somewhat different from what Popper wanted to use probability for, but since his axiomatization is quite formal, Field can use it in a different interpretation.) We can thus think of the probability as in some sense giving the semantics of the language. If an agent has $P(A|B) = 1$, then she is completely willing to make the inference to A from B. If this inference is licensed by all agents with degrees of belief satisfying the axioms, then this inference is in some sense valid as a matter of logic. So if we can justify Popper's axioms as playing the sort of inferential role that Dummett wanted to base logic on, then this gives a justification of classical logic, rather than the restricted intuitionist logic Dummett preferred. (However, see section 5 for more discussion on this point.)

Since Popper's axioms only deal with the sentential connectives, and not with quantifiers and predicates, Field gives a way to extend the axioms to deal with the quantifiers. This is important for the project he wants, because part of the distinctive character of intuitionist logic is carried by the fact that the existential and universal quantifiers are not dual, in addition to the behavior of the connectives. But this extension is more problematic and involves quantifying over multiple probability functions rather than just using a single function.

4.2 Adams

Another important attempt to derive a version of logic from probability theory is pursued in the work of Ernest Adams. (See (Adams 1998) for a thorough discussion, but the ideas are present in many papers of his as well.) I have mentioned above a notion of logical validity on which the argument from B to A is valid iff $P(A|B) = 1$ for all probability functions satisfying a certain class of axioms. Field also mentions another notion of validity, on which the argument is valid iff $P(A|C) \geq P(B|C)$ for all probability functions and all C. Adams proposes yet another notion of validity, on which an argument is valid iff for all probability functions, and every ϵ there is a δ, such that if all premises have probability at least $1 - \delta$, then the conclusion must have probability at least $1 - \epsilon$.

For most ordinary statements, and standard probability functions, these three conditions are equivalent. But Adams proposes his new condition in order to extend the logic to a type of statements that he suggests haven't been treated appropriately by classical logic, namely conditionals.

It is well-known that the logic of the material conditional (which is the conditional standardly discussed in classical truth-functional logic) seems to diverge in important ways from the logic of the ordinary English-language indicative conditional. Although $A \supset B$ follows from $\neg A$, it doesn't seem appropriate to conclude "If A then B" on the basis of $\neg A$.

Adams suggests that we think of the conditional $A \rightarrow B$ as a proposition that is not given as a truth-functional compound of A and B. Instead, we should think of it as something distinct from a proposition, but it still has a probability, given by the relevant conditional probability. Thus, he defines $P(A \rightarrow B) = P(B|A)$, even though he suggests that "$A \rightarrow B$" isn't really a proposition.

Given this interpretation, we can see that the above-mentioned inference is not validated on Adams' conception of logic. No matter how high $P(\neg A)$ is, provided that it is different from 1, it is easy to find a probability function on which $P(B|A)$ is still quite low. (For instance, we can just consider any probability function on which $P(B) = P(A)/100$.) However, the logic still validates traditional rules of inference like modus ponens and modus tollens.[3]

The major worry for this notion of logic is that the identification of the probability of a conditional with a conditional probability causes serious troubles. Starting with David Lewis and extending through the work of authors like Ned Hall, Alan Hájek, and others, there have been a series of triviality results. (Lewis 1976), (Hall 1994), (Hájek 1994) These results show

[3]For modus ponens, note that if $P(A) > \sqrt{1-\delta}$ and $P(B|A) > \sqrt{1-\delta}$, then $P(B) \geq P(A \wedge B) = P(A)P(B|A) = 1 - \delta$. For modus tollens, note that $P(A) = P(A \wedge B)/P(B|A)$. Thus, if $P(A \wedge B) < \epsilon$ (as is the case if $P(\neg B) > 1 - \epsilon$) and $P(B|A) > 1 - \epsilon$, then $P(A) < \epsilon/(1-\epsilon)$. By choosing ϵ small enough, we can clearly guarantee that this is below δ.

that if the probability of a conditional is given by the conditional probability, and the conditional satisfies certain other assumptions in general, then the theory is trivial in some way. The mildest results just show that the conditional must be logically equivalent to the material conditional (which would defeat the purpose), while the strongest results show that the probability function must never take on any value other than 0 or 1! Clearly, such a theory would hardly deserve the name of probability.

The response by Adams and his defenders has been to reject all these further assumptions about conditionals. In particular, it must not be the case that $(A \wedge B) \to C$ is equivalent to $A \to (B \to C)$. A common claim has been that conditionals don't embed at all (so that the latter claim is in some sense meaningless), and that conditionals are not the ordinary sort of propositions that have truth-values at a world. Instead, conditionals are used to merely express some claim about an agent's conditional degrees of belief. But the details of these theories are too intricate to do them justice in this brief passage. For more on these sorts of theories of the conditional, see (Bennett 2003) and (Edgington 1995).

5 Probability and non-classical logics

Finally, I come to the topic of non-classical logic. (Weatherson 2003) gives a slight generalization of the probability axioms that he suggests might work better for dealing with non-classical logics:

1. If A entails every statement in the language, then $P(A) = 0$.
2. If A is entailed by every statement in the language, then $P(A) = 1$.
3. If $A \vdash B$, then $P(A) \leq P(B)$.
4. $P(A) + P(B) = P(A \vee B) + P(A \wedge B)$.

The third axiom here is the same as the corresponding standard axiom, but the first standard axiom is split into two cases, and the additivity axiom is slightly generalized. In cases where A and B are contradictory, we see that the first and fourth axioms here give the standard additivity axiom. But this axiom applies even in cases where they are not contradictory.

Weatherson then suggests two motivations for using this slightly modified axiom scheme, together with intuitionist logic. One reason is that one might be motivated by the concerns of Dummett and others to make sure that the logic underlying one's language is intuitionist. (Of course, this would mean rejecting Field's response to Dummett mentioned above.)

But he gives another reason for adopting this sort of intuitionist logic, which is based on worries about the connection between evidence and degrees of belief. One might, for instance, suppose that if one has no evidence for a claim A, then one should have degree of belief 0 in the claim. But in many situations, one has no evidence either for a claim or its negation. Weatherson points out that some authors have used this concern to argue against standard probability theory in favor of some non-standard alternative (for instance, (Cohen 1977)), but he points out that this can be accommodated by moving to his slight modification of probability while adopting an intuitionist logic.

Weatherson also suggests that this axiomatization can be used to develop probability theory based on other non-classical logics, like that of Łukasiewicz, but this suggestion is developed in much less detail.

6 Conclusion

There are many topics in the relation of probability to logic that I have not had space to discuss here, but these are some that I have found the most interesting. Several of these issues are discussed in greater depth in other discussions of the topic, like (Roeper & LeBlanc 1999) (which goes into much greater depth on issues related to the work of Adams and Popper, as well as discussing probability semantics for languages with infinitary operations, and an alternate way to deal with intuitionist logic), (Hájek 2001) (which gives much more thorough discussion to the various interpretations of probability, as well as some competitors to probability theory), and (Earman 1992) (which discusses the issues surrounding Bayesianism in the philosophy of science, though largely not in the guise of a type of "inductive logic").

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